Question Sheet 4, Continuity I v1 2020-21

Continuous functions

Six examples of showing a function is continuous.

1. Let
$$f(x) = \frac{x^2 - 2x - 15}{x+3}, x \neq -3.$$
How should $f(-3)$ be defined so that f is continuous at -3?

2. Prove, by verifying the ε - δ definition that $h(x) = |x|$ is continuous at $x = 0$.

   Deduce that h is continuous on $\mathbb{R}$.

   (You need not verify the definition for $x \neq 0$, instead quote results from the lecture notes.)

3. Prove, by verifying the ε - δ definition that

   i) the function $f(x) = x^2$ is continuous on $\mathbb{R}$,

   Hint Look back at Question 2 on Question Sheet 1 and replace $a = 2$ seen there by any $a \in \mathbb{R}$.

   ii) the function $g(x) = \sqrt{x}$ is continuous on $(0, \infty)$.

   Hint Look back at Question 11 on Question Sheet 1 and replace the $a = 9$ seen there by any $a > 0$.

   iii) the function
$$h(x) = \begin{cases} x^2 + x & \text{for } x \leq 1 \\ \sqrt{x+3} & \text{for } x > 1, \end{cases}$$
   is continuous at $x = 1$.

   Hint Verify the ε - δ definitions of both one-sided limits separately at $x = 1$.

   iv) the function
$$\frac{1}{x^2+1}$$
   is continuous on $\mathbb{R}$.

4. Are the following functions continuous on the domains given or not?

   Either prove that they are continuous *by using the appropriate Continuity Rules*, or show they are not.

   i)
   $$f(x) = \frac{x+2}{x^2+1} \text{ on } \mathbb{R}.$$

   ii)
   $$g(x) = \frac{3+2x}{x^2-1},$$

   firstly on $[-1/2, 1/2]$, secondly on $[-2, 2]$.

   iii)
   $$h(x) = \frac{x^2+x-2}{(x^2+1)(x-1)} \text{ on } \mathbb{R}.$$

   iv)
   $$j(x) = \begin{cases} x+2 & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x \le 1 \\ x-2 & \text{if } x > 1. \end{cases}.$$

   v)
   $$k(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0. \end{cases}.$$

   vi)
   $$\ell(x) = \begin{cases} \dfrac{1-\cos x}{x^2} & x \neq 0 \\ 1 & x = 0. \end{cases}.$$

5. i) Prove, by verifying the definition, that $\cos x$ is continuous on $\mathbb{R}$.
   Hint Make use of $\cos(x+y) = \cos x \cos y - \sin x \sin y$, valid for all $x, y \in \mathbb{R}$.

   ii) Prove that $\tan x$ is continuous for all $x \neq \pi/2 + k\pi, k \in \mathbb{Z}$.

6. Show that the hyperbolic functions $\sinh x$, $\cosh x$ and $\tanh x$ are continuous on $\mathbb{R}$.

Composite Rule

7. i) State the Composite Rule for functions.

   Evaluate
$$\lim_{x \to 0} \exp\left(\frac{\sin x}{x}\right).$$

   ii) State the Composite Rule for continuous functions.

   Prove that
$$\left|\frac{x+2}{x^2+1}\right|$$
   is continuous on $\mathbb{R}$.

Intermediate Value Theorem

8. State the Intermediate Value Theorem.

   Give an example of a strictly increasing function f on $[0,1]$ and a value $\gamma : f(0) < \gamma < f(1)$ for which there is **not** a $c \in [0,1]$ with $f(c) = \gamma$.

9. Show that
$$e^x = \frac{1}{x}$$
   has **a** solution in $[0,1]$.

10. Show that $e^x = 4x^2$ has **at least three** real solutions.